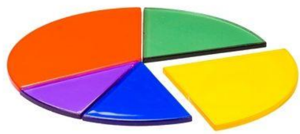
6. REAL AND COMPLEX NUMBERS

§6.1. Fractions



The next stage in our development of the complex number system is to develop fractions or positive rational numbers. This mirrors the historical development of numbers because negative numbers came much later than fractions.

The concept of a fraction is quite sophisticated and it's no wonder that school students have so much trouble with them. For a start, what is a fraction? If you can do no better than talk about cutting up a cake into equal parts then your knowledge of fractions is somewhat underdeveloped.

At first glance a fraction is a pair of natural numbers. But then we talk about **equivalent** fractions. Different pairs can represent the same number, as in $\frac{6}{8} = \frac{3}{4}$. So fractions are more complicated than number pairs.

For a start we' exclude zero. We'll bring it in at a later stage. So we begin by taking $M = \mathbb{N} - \{0\}$ to be the set of non-zero natural numbers. Then we form $M \times M$, the set of ordered pairs (m, n) of non-zero natural numbers.

Now we define the relation ~ on this set: $(a, b) \sim (c, d)$ if $ad = bc$.

The first thing to do is to check that this is an equivalence relation, that is, it is reflexive, symmetric and transitive.

Theorem 1: The relation ~ is an equivalence relation.
Proof: Reflexive: $(a, b) \sim (a, b)$ since $ab = ba$.
Symmetric: Suppose $(a, b) \sim (c, d)$.
Then $ad = bc$ and so $cb = da$.
Hence $\{c, d) \sim (a, b)$.
Transitive: Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.
Then $ad = bc$ and $cf = de$.
Hence $(ad)(cf) = (bc)(de)$
$\therefore$ $(af)(cd) = (be)(cd)$
$\therefore$ $af = be$ (Remember that c, d are non-zero.)
$\therefore$ $(a, b) \sim (e, f)$. ✋☺

The set M × M thus decomposes into equivalence classes and these are our fractions. We'll use the term fraction for what we'd normally call a positive rational number. Negative rational numbers will come later.

We denote the equivalence class containing the ordered pair (m, n) by the familiar symbol $\frac{m}{n}$.
Possibly we should have written the equivalence classes as $[m, n]$ instead of $\frac{m}{n}$. The danger is to use things you learnt in primary school instead of justifying them. So it

is true that $\frac{6}{8} = \frac{3}{4}$, for example, but not because we cancel top and bottom by 2, but because $6 \times 4 = 3 \times 8$.

Now we must define addition and multiplication, not of the ordered pairs, but of the equivalence classes. This involves what are called "checks of well-definedness".

We define $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$. This is something that any high-school student would recognise. But remember these fractions are equivalence classes of ordered pairs. We have now to check that this is a proper definition. The same fraction can be expressed using different natural numbers. There is the possibility of getting different answers by choosing different representations.

Example 1: $\frac{10}{15} + \frac{3}{7} = \frac{70 + 45}{105} = \frac{115}{105}$ and

$$\frac{2}{3} + \frac{6}{14} = \frac{28 + 18}{42} = \frac{46}{42}.$$

Yet $\frac{10}{15} = \frac{2}{3}$ and $\frac{3}{7} = \frac{6}{14}$ so the answers should be the same. They look different, and in fact as ordered pairs they *are* different.

But $\frac{115}{105}$ is equivalent to $\frac{46}{42}$ since

$$115 \times 42 = 4830 = 105 \times 46.$$

Of course you would probably have cancelled common factors of each fraction, and this is easily justified. So that we would say that $\frac{115}{105} = \frac{23}{21} = \frac{46}{42}$.

Probably this would have once convinced you. But now you should be saying that at least sometimes the definition works properly. But does it *always* work? To prove that it does we repeat the above exercise with general fractions.

Theorem 2: Addition of fractions is well-defined.

Proof: Suppose $\frac{a'}{b'} = \frac{a}{b}$ and $\frac{c'}{d'} = \frac{c}{d}$.

Then $a'b = b'a$ and $c'd = d'c$.

Now $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ and

$$\frac{a'}{b'} + \frac{c'}{d'} = \frac{a'd' + b'c'}{b'd'}.$$

We need to check that:

$$(ad + bc)(b'd') = (a'd' + b'c')(bd).$$

The LHS $= adb'd' + bcb'd'$

$= (b'a)(dd') + (d'c)(bb')$

$= (a'b)(dd') + (c'd)(bb')$

$= (a'd')(bd) + (b'c')(bd)$

$=$ RHS. ✋☺

We define the product of two fractions by $\frac{a}{b} . \frac{c}{d} = \frac{ac}{bd}$.
Again we need to check that this is well-defined.

Theorem 3: Multiplication of fractions is well-defined.

Proof: Suppose $\frac{a'}{b'} = \frac{a}{b}$ and $\frac{c'}{d'} = \frac{c}{d}$.

Then $a'b = b'a$ and $c'd = d'c$.

Now $\frac{a}{b} . \frac{c}{d} = \frac{ac}{bd}$ and $\frac{a'}{b'} . \frac{c'}{d'} = \frac{a'c'}{b'd'}$.

Finally, $(ac)(b'd') = (b'a)(d'c) = (a'b)(c'd) = (a'c')(bd)$, so the products are the same. ✋☺

§6.2. The Arithmetic of Fractions

We now have to prove all the familiar properties of the positive rational numbers. These make constant use of the corresponding laws for natural numbers.

Theorem 4: Addition and multiplication of fractions is commutative and associative.

Proof: The commutative laws for addition and multiplication are obvious from the symmetry of the definitions. The associative laws are a little less obvious.

Associativity of Addition:

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{ad + bc}{bd} + \frac{e}{f} = \frac{(ad + bc)f + bde}{bdf}.$$

$$\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right) = \frac{a}{b}+\frac{cf+de}{df} = \frac{adf+b(cf+de)}{bdf}.$$

These answers are equal because:

$$(ad+bc)f+bde = adf+b(cf+de).$$

Associativity of Multiplication:

$$\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f} = \frac{ac}{bd}\cdot\frac{e}{f} = \frac{ace}{bdf}.$$

$$\frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{e}{f}\right) = \frac{a}{b}\cdot\frac{ce}{df} = \frac{ace}{bdf}.$$

Here the ordered pairs are identical, not just equivalent. ✋☺

Theorem 5: Multiplication of fractions is distributive over addition.

Proof: $\frac{a}{b}\cdot\left(\frac{c}{d}+\frac{e}{f}\right) = \frac{a}{b}\cdot\left(\frac{cf+de}{df}\right) = \frac{a(cf+de)}{bdf}.$

$$\frac{a}{b}\cdot\frac{c}{d}+\frac{a}{b}\cdot\frac{e}{f} = \frac{ac.bf+bd.ae}{b^2df}.$$

In this case the ordered pairs are different. You probably would have justified the equality of these expressions by 'cancelling by *b*', and that can be justified. However we observe that $(acf+ade).b^2df = bdf\,(acbf+bdae)$. ✋☺

As we've constructed them, the set of positive natural numbers is not a subset of the set of fractions. The former are finite sets while the latter are equivalence classes of pairs of natural numbers and hence are infinite sets. But we identify the positive natural number *n* with the fraction $\frac{n}{1}$, which in turn consists of all the ordered pairs (kn, k) for all non-zero natural numbers k. But to justify this we'd to check that the fractions of the form $\frac{n}{1}$ behave like the natural numbers n themselves.

$$\frac{m}{1} + \frac{n}{1} = \frac{m.1 + 1.n}{1.1} = \frac{m + n}{1} \text{ and}$$

$$\frac{m}{1} . \frac{n}{1} = \frac{m.n}{1.1} = \frac{mn}{1} .$$

So the system of fractions contains within in it a working model of the natural numbers.

Example 2: The numbers 2, as a natural number, is different to the fraction $\frac{2}{1}$ even though it behaves in the same way as both and even though we happily blur the distinction in normal mathematics.

As a natural number $2 = \{\varnothing, \{\varnothing\}\}$.

As a fraction 2 is $\frac{2}{1} = \{(2k, k) \mid k \in \mathbb{N} - \{0\}\}$

$$= \{(2, 1), (4, 2), (6, 3), \ldots\}$$

Just the first element in this list is

$(2, 1) = \{\{2\}, \{1, 2\}\}$

$= \{\{\{\varnothing, \{\varnothing\}\}\}, \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}$.

Division of fractions is now defined in the usual way – invert and multiply – and the usual properties of division are now easily verified.

§6.3. The Order Relation for Fractions

We now define the ordering of the fractions.

We define $\frac{a}{b} \le \frac{c}{d}$ if $ad \le bc$.

Theorem 6: The order relation is well-defined.

Proof: Suppose $\frac{a'}{b'} = \frac{a}{b}$ and $\frac{c'}{d'} = \frac{c}{d}$.

Then $a'b = b'a$ and $c'd = d'c$.
We must show that $ad \le bc$ if and only if $a'd' \le b'c'$.
Suppose $ad \le bc$.
Then $(ad)(a'd') \le (bc)(a'd')$.
But $(bc)(a'd') = (a'b)(d'c) = (ab')(dc') = (ad)(b'c')$.
Hence $(ad)(a'd') \le (ad)(b'c')$ and so $a'd' \le b'c'$. ✋☺

§6.4. Positive Real Numbers

Having constructed fractions, that is, the positive rational numbers, we now turn our attention to creating the positive real numbers. We can't define these as numbers that represent points on the positive part of the real axis because this requires geometric intuition. We could define them in terms of decimal expansions, though this would become clumsy when it came to defining addition and multiplication. They are often defined as limits of convergent sequences of rational numbers. In keeping with our principle that every mathematical object has to be a set we'll define them as sets of rational numbers with certain properties.

A relation R is **anti-symmetric** if $x\text{R}y \wedge y\text{R}x \rightarrow x = y$.

A **partial order** on a set S is a relation that is reflexive, anti-symmetric and transitive.

A **partially ordered set** is a set together with a partial order $\leq$.

An **upper bound** for a subset X of S, if one exists, is an element $u \in S$ such that

$$\forall x[x \in \text{X} \rightarrow x \leq u].$$

A **greatest element** for X, if it exists, is an upper bound for X that is an element of X. Clearly, if it exists, it is unique.

A subset X is an **initial segment** if

$$\forall \text{x}\forall \text{y}[(x \in \text{X}) \wedge (y < x) \rightarrow (y \in \text{S})].$$

Example 3: On the set $\mathbb{Q}^+$ of positive rational numbers
$\text{X}_1 = \mathbb{Q}^+$ has no upper bound;
$\text{X}_2 = \mathbb{N}$ has no upper bound;
$\text{X}_3 = \{x \in \mathbb{Q}^+ \mid x^2 < 2\}$ has upper bounds, such as 2, and 200, but it doesn't have a greatest element.
$\text{X}_4 = \{x \mid x^2 + 6 = 5x\}$ has a greatest element, namely 3. (The quadratic $x^2 - 5x + 6$ has two zeros, 2 and 3.)
$\text{X}_5 = \varnothing$ has upper bounds (in fact every element) but no greatest element.

$X_6 = \{x \mid x^2 \le 4\}$ has upper bounds and a greatest element.

X_1, X_3, X_5 and X_6 are initial segments.

A **positive real number** is a non-empty initial segment of $\mathbb{Q}^+$ that has an upper bound but no greatest element.

Since a positive real number is a subset of $\mathbb{Q}^+$, it is a set. And the class of all positive real numbers is a subset of $\wp\mathbb{Q}^+$ and so is a set. We denote this by the symbol $\mathbb{R}^+$.

Example 4: Of the above subsets of $\mathbb{Q}^+$ only X_3 qualifies as a positive real number. This we'll identify with the real number $\sqrt{2}$. X_1 has no upper bound, X_5 is empty and X_6 has a greatest element.

If X, Y are positive real numbers we define X + Y, XY and X ≤ Y in terms of their elements as follows:

$$X + Y = \{x + y \mid x \in X, y \in Y\};$$

$$XY = \{xy \mid x \in X, y \in Y\}.$$

$$X \le Y \text{ if and only if } X \subseteq Y.$$

Theorem 7: If X, Y are positive real numbers then so are X + Y and XY.

Proof: Suppose that X, Y are positive real numbers. Clearly X + Y and XY are non-empty.

If u is an upper bound for X and v is an upper bound for Y then $u + v$ and uv are clearly upper bounds for X + Y and XY respectively.

Let $x + y \in X + Y$.
Since X, Y have no maximum elements, there exist $x_1 \in X$ and $y_1 \in Y$ such that $x < x_1$ and $y < y_1$.
Hence $x + y < x_1 + y_1 \in X + Y$.
Similarly for multiplication.
Finally we must show that X + Y and XY are initial segments.
Let $u = x + y$ where $x \in X$ and $y \in Y$ and let $v < u$.

Multiplication is a little trickier. Let $u = xy$ and let $w = u - v > 0$.
We want to find x_1 and y_1 such that $v = x_1y_1$ and $0 < x_1 < x$ and $0 < y_1 < y$.
Write $x_1 = x - a$ and $y_1 = y - b$.

$$\begin{aligned} \text{We want } v &= (x - a)(y - b) \\ &= xy - (ay + bx) + ab \\ &= u - (ay + bx) + ab \end{aligned}$$

so $w = ay + bx - ab$. ✋☺

If we chose any b and defined $a = \dfrac{w - bx}{y - b}$ then:
$ay + bx - ab$ would equal to w and so, working backwards, $v = (x - a)(y - b) \in XY$.

This would be fine over $\mathbb{Q}$, provided $b \neq y$ but since we are working over $\mathbb{Q}^+$ we must ensure that $w - bx$, $x - a$ and $y - b$ are all positive. The fact that v and $y - b$ are positive will ensure that $x - a$ is positive so we only need to worry about $w - bx$ and $y - b$.

We can achieve this by taking $b = \text{MIN}\left(\frac{w}{2x}, \frac{y}{2}\right)$ and

$a = \dfrac{w - bx}{y - b}$.

We can verify that indeed $v = (x - a)(y - b) \in \text{XY}$.

The usual associative, commutative and distributive properties can easily we deduced from the corresponding properties for positive rational numbers. An important property of the real numbers is completeness – something that distinguishes it from the rational numbers.

Theorem 8 (COMPLETENESS): Every non-empty subset of the rational numbers that has an upper bound has a least upper bound.

Proof: At this stage we can only prove it for the positive reals.

Suppose that Σ is such a non-empty subset of $\mathbb{R}^+$ that is bounded above.

We have to work with sets at several levels. Positive rationals will be denoted by lower case letters. Positive real numbers are sets of positive rationals and will be written in upper case. Finally Σ is a set of positive reals, which is why we use a Greek symbol. I hope this will help to make it all clear!

We must prove that $\cup\Sigma$ is a positive real. So in anticipation we will denote it by the symbol S.

S = $\cup\Sigma$ is non-empty:

Let X $\in$ Σ (OK as Σ is non-empty).

X is a positive real number and so, by definition, it is non-empty.

Let $x \in$ X. Then $x \in \cup\Sigma$.

S = $\cup\Sigma$ is bounded above:

Σ is bounded above.

Let M be an upper bound for Σ. This means that every element of Σ is less than or equal to *M*.

Now M is a positive real and so is a set of positive rationals.

By definition it is bounded above, say by the rational number *m*.

Let $x \in$ S = $\cup\Sigma$. Then $x \in$ X for some X $\in \Sigma$.

Since M is an upper bound for Σ, X $\le$ M, in other words, X $\subseteq$ M.

Hence $x \in$ M. Since *m* is an upper bound for M, $x \le m$.

It follows that *m* is an upper bound for S = $\cup\Sigma$.

S = $\cup\Sigma$ is an initial segment:

Let $x \in$ S and let $y \le x$.

Then $x \in$ X for some X $\in$ S.

Now, being a positive real, X is an initial segment and so $y \in$ X.

Hence $y \in$ S = $\cup\Sigma$,

This completes the proof that $\cup\Sigma$ is a positive real number.

S = ∪Σ is an upper bound for Σ.
Let X ∈ Σ. Then X ⊆ S.
But, for positive reals, this is the same as less than or equals. That is, X ≤ S.

S = ∪Σ is the least upper bound for Σ.
Suppose that T is an upper bound for Σ.
Let x ∈ S. Then x ∈ X for some X ∈ Σ.
Since T is an upper bound for Σ, X ≤ T.
But X, T are positive reals and so this translates to
X ⊆ T. Hence x ∈ T.
So S ⊆ T, which translates to S ≤ T.
So S is the least upper bound for Σ. ✋☺

We can now show that for a positive rational number q the corresponding positive real number:

$$\{x \mid x < q\}$$

behaves exactly like the positive rational number itself and so we associate q with $\{x \mid x < q\}$. Within the positive real numbers we have a copy of the positive rational numbers.

Hence the multiplicative identity of the positive reals is $\{x \mid x < 1\}$ which we associate with the rational number 1. And the positive real $\{x \mid x^2 < 2\}$ has the property that its square is $\{x \mid x < 2\}$ which we associate with the rational number 2. So we denote $\{x \mid x^2 < 2\}$ by $\sqrt{2}$ and note that indeed it is the (positive) square root of 2.

We saw that the fraction 2, as we have defined it, is rather more complicated, when expressed in terms of the empty set, than the natural number 2. The real number 2 is even worse! It's the set of all fractions $\frac{a}{b}$ where $a < 2b$. Just one of these elements is $\frac{3}{2}$. This in turn is the set of all ordered pairs of the form $(3k, 2k)$. One of these elements is (3, 2) which is {{3}, {2, 3}}. Finally, one of these two elements is {2, 3} which is {{∅, {∅}}, {∅, {∅}, {∅, {∅}}}}.

But, as before, we don't have to be embroiled in these complexities. It's sufficient for us to be aware that everything we construct can be built up from the empty set. But whenever we have occasion to use the number 2 in a piece of mathematics we don't have to worry whether it is the real number 2, the rational number 2, or the natural number 2. They are all identified with one another and can be treated as if they are the same thing.

There are many other properties that ought to be investigated but that is done in a standard course on analysis. Many of these follow from the completeness of the positive reals which we have proved. But now we move on to negative numbers.

§6.5. Zero and Negative Real Numbers

The construction that takes us from the positive real numbers to the whole real line is similar to the one that that took us from the natural numbers to the positive rationals. We take pairs of real numbers and define an equivalence relation. The real numbers are simply the equivalence classes.

The pair $\langle a, b\rangle$ will ultimately represent $a - b$ which, depending on the relative sizes of a, b can give positive, negative or zero real numbers. But because the representation as $a - b$ is not unique we must take equivalence classes.

So take the set $\mathbb{R}^+ \times \mathbb{R}^+$ to be the set of all pairs of positive real numbers and define the relation $\equiv$ as follows:

$$(a, b) \equiv (c, d) \text{ if and only if } a + d = b + c.$$

I leave it as an exercise to prove that this is indeed an equivalence relation. Let $\langle x, y\rangle$ denote the equivalence class containing (x, y). So if $a + d = b + c$ then

$$\langle a, b\rangle = \langle c, d\rangle.$$

Define addition, multiplication and ordering by:

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle;$$

$$\langle a, b\rangle . \langle c, d\rangle = \langle ac + bd, ad + bc\rangle;$$

$$\langle a, b\rangle \le \langle c, d\rangle \text{ if } a + d \le b + c.$$

These definitions are motivated by the fact that:

$(a - b) - (c - d) = (a + c) - (b - d)$,

$(a - b).(c - d) = (ac + bd) - (ad + bc)$ and

$a - b \le c - d$ if and only if $a + d \le b + c$.

You know what lies ahead. We must show that these operations are well-defined and that the associative, commutative properties hold as well as all the other elementary properties. We don't need to actually do this, except as an exercise. The main thing is to be convinced that *it can be done*!

We might be tempted to identify the positive real numbers with those real numbers of the form $[x, 0]$ but, of course 0 is not positive. Instead we must identify $\langle x + 1, 1\rangle$ with x.

Theorem 9: $F:R^+ \to R$ defined by $F(x) = \langle x + 1, 1\rangle$ takes sums to sums and products to products.

Proof:

$$
\begin{aligned}
F(x) + F(y) &= \langle x + 1, 1\rangle + \langle y + 1, 1\rangle \\
&= \langle x + y + 2, 2\rangle \\
&= \langle x + y + 1, 1\rangle \text{ since} \\
&\qquad (x + y + 2) + 1 = (x + y + 1) + 2 \\
&= F(x + y).
\end{aligned}
$$

$$
\begin{aligned}
F(x)F(y) &= \langle x + 1, 1].\langle y + 1, 1\rangle \\
&= \langle (x + 1)(y + 1) + 1, (x + 1) + (y + 1)\rangle \\
&= \langle xy + x + y + 2, x + y + 2\rangle \\
&= \langle xy + 1, 1\rangle \\
&= F(xy).
\end{aligned}
$$

§6.6. Complex Numbers

We're almost there. The last stage in the development of our number system is to extend the real numbers to the system of complex numbers, ℂ. This is by far the easiest stage of all. We define a complex number to be a pair of real numbers. There is no equivalence relation, and no equivalence classes needed. Complex numbers are just the pairs themselves, which we will write as $[x, y]$ instead of the usual (x, y) to make it look more like the previous stages.

A complex number is a pair of real numbers $[x, y]$. Keeping in mind that we'll eventually identify this with $x + \mathrm{i}y$, where $\mathrm{i}^2 = -1$, we make the following definitions:

$[a, b] + [c, d] = [a + c, b + d]$;
$[a, b] \,.\, [c, d] = [ac - bd, ad + bc]$;

Note that we don't define an ordering. There is no order relation that's consistent with these algebraic operations.

We identify the complex numbers of the form $[x, 0]$ with the corresponding real numbers x and note that they behave the same.

$[x, 0] + [y, 0] = [x + y, 0 + 0] = [x + y, 0]$ and
$[x, 0].[y, 0] = [xy - 0, 0 + 0] = [xy, 0]$.

The multiplicative identity is [1, 0], which we identify with the real number 1 and we define i = [0, 1]. Note that $i^2 = [-1, 0]$ which we identify with –1.

The fact that [*x*, *y*] = [*x*, 0] + [0, 1].[*y*, 0] means we can identify [*x*, *y*] with *x* + i*y*.

All that remains is to check out the basic properties of complex numbers and we're finished. But probably by now you're thoroughly bored with the whole process and are quite happy in just accepting the assurance "believe me, it works!"

Throughout all this construction, the various types of numbers are sets and sets of sets and sets of sets etc, all built out of the empty set. We started with the natural numbers and extended it to the positive rationals, then to the positive reals, then to all the reals and finally to the complex numbers.

$$\mathbb{N} \rightarrow \mathbb{Q}^{+} \rightarrow \mathbb{R}^{+} \rightarrow \mathbb{R} \rightarrow \mathbb{C}$$

It is interesting that this corresponds to the order in which these number systems were developed historically.

As we've constructed them these sets are not subsets of one another. But within each we can identify a subset that we can identify with the previous one, and so we consider these to be nested as follows:

$$\mathbb{N} \subset \mathbb{Q}^{+} \subset \mathbb{R}^{+} \subset \mathbb{R} \subset \mathbb{C}$$

Example 5: The complex number 2 is
$[2, 0] = \{\{2\}, \{2, 0\}\}$ where 0, 2 are in $\mathbb{R}$.
The real number 2 is $\{(3 + k, 1 + k) \mid k \in \mathbb{R}^+\}$
$= \{\{3 + k\}, \{1 + k, 1 + k\}\}$ where 1, 3. k are in $\mathbb{R}^+$.
The positive real number 3 is the set $\{x \in \mathbb{Q}^+ \mid x < 3\}$ where 3 is in $\mathbb{Q}^+$.
The positive rational number 3 is $\frac{3}{1}$ which is
$\{(3k, k) \mid k \in \mathbb{N}\} = \{\{\{3k\}, \{3k, k\}\} \mid k \in \mathbb{N}\}$
where 3, k are in $\mathbb{N}$.
The natural number 3 is $\{0, 1, 2\}$
$= \{\varnothing, \{\varnothing\}, \{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}$.

To give a map of where we've come, here are the universes in which each number lives. Remember that sets of elements of S are in $\wp S$ and pairs of elements of S are in $\wp^2 S$ and equivalence classes of pairs of elements of S are in $\wp^3 S$.

Numbers	**are elements of**
natural numbers	$\mathbb{N}$
positive rationals	$\wp^3\, \mathbb{N}$
positive reals	$\wp^4\, \mathbb{N}$
reals	$\wp^7\, \mathbb{N}$
complex numbers	$\wp^9\, \mathbb{N}$